Differential Geometry I: Curves and surfaces

11.1 Let $\gamma : [a, b] \to \mathbb{R}^3$ be a Frenet-regular curve and let $\varepsilon > 0$ be a (small) constant. The union of the circles of radius ε centered at $\gamma(s)$ and contained in the plane orthogonal to $\dot{\gamma}(s)$ is a surface. We call it the ε-tube around γ (thus a cylinder or a torus are simple examples of tubes).

(a) Assuming that γ is arc-length parametrized and biregular, give a parametrization $\psi(s, \theta)$ of the ε-tube (use the Frenet frame).

(b) Compute the metric tensor of this parametrization.

(c) Show that the area of this tube is given by

$$A = 2\pi\varepsilon L,$$

where L is the length of γ. Observe that this formula is surprising: the area of the tube depends only on ε and the length of the centerline curve γ. Nevertheless give an intuitive explanation for this phenomenon.

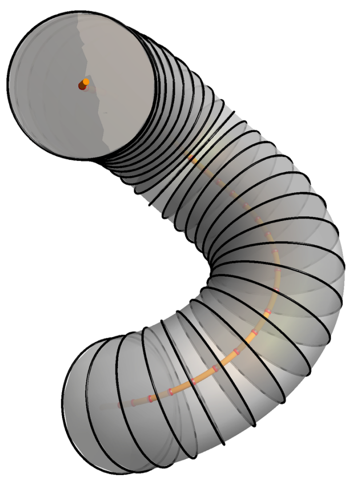


Figure 1: An example of a tubular surface (around the centerline orange curve).

Solution.

(a) A parametrization of the ε-tube is

$$\psi(s, \theta) = \gamma(s) + \varepsilon \cos\theta \; N(s) + \varepsilon \sin\theta \; B(s),$$

where $\{T, N, B\}$ is the Frenet frame of γ, and the parameter domain is $\Omega := \{(s, \theta) \mid a \leqslant s \leqslant b, \; 0 \leqslant \theta \leqslant 2\pi\}$.

(b) The adapted frame of the parametrization is

$$\begin{aligned} b_1(s,\theta) = \frac{\partial \psi}{\partial s} &= \dot{\gamma}(s) + \varepsilon \cos\theta\, \dot{N}(s) + \varepsilon \sin\theta\, \dot{B}(s) \\ &= T + \varepsilon\cos\theta(-\kappa(s)T(s) + \tau(s)B(s)) - \varepsilon\sin\theta\,\tau(s)N(s) \\ &= (1-\varepsilon\kappa(s)\cos\theta)T(s) - \varepsilon\tau(s)\sin\theta\, N(s) + \varepsilon\tau(s)\cos\theta\, B(s), \end{aligned}$$

$$b_2(s,\theta) = \frac{\partial \psi}{\partial \theta} = -\varepsilon \sin\theta\, N(s) + \varepsilon\cos\theta\, B(s).$$

(We used the Serret–Frenet formulas.) The metric tensor $G = (g_{ij})$ with $g_{ij} = \langle b_i, b_j\rangle$ is

$$G(s,\theta) = \begin{pmatrix} (1-\varepsilon\kappa(s)\cos\theta)^2 + (\varepsilon\tau(s))^2 & \varepsilon^2\tau(s) \\ \varepsilon^2\tau(s) & \varepsilon^2 \end{pmatrix}.$$

In particular

$$\sqrt{\det G(s,\theta)} = \varepsilon(1-\varepsilon\kappa(s)\cos\theta).$$

Note that although the metric depends on curvature and torsion, the determinant depends only on the curvature κ.

(c) The area is

$$\begin{aligned} A &= \int_a^b \int_0^{2\pi} \sqrt{\det G(s,\theta)}\, d\theta\, ds = \int_a^b \int_0^{2\pi} \varepsilon(1-\varepsilon\kappa(s)\cos\theta)\, d\theta\, ds \\ &= \int_a^b 2\pi\varepsilon\, ds = 2\pi\varepsilon(b-a) = 2\pi\varepsilon L, \end{aligned}$$

since $\int_0^{2\pi}\cos\theta\, d\theta = 0$.

Intuitively an ε-tube behaves like a flexible pipe: when you bend it some parts are stretched and others are compressed. These local area changes compensate globally so that the total area depends only on the tube radius and on the length of the centerline.

11.2 Let $\gamma : I \to \mathbb{S}^2$ be a simple C^1 curve drawn on the unit sphere; assume γ is arc-length parametrized. Consider the cone C with vertex 0 generated by this curve, i.e. the set of half-lines starting at 0 and passing through a point of γ.

- **(a)** Give a parametrization of C as a ruled surface and show that $C \setminus \{0\}$ is a submanifold of $\mathbb{R}^3$.
- **(b)** Compute the metric tensor for this parametrization.
- **(c)** Show that $C \setminus \{0\}$ is locally isometric to the Euclidean plane.

Solution. (a) Since $\|\gamma(u)\| = \|\dot{\gamma}(u)\| = 1$ for all u and γ is injective (the curve is simple), each half-line through $\gamma(u)$ is parametrized by $v \mapsto v\gamma(u)$ ($v \geqslant 0$). Thus a parametrization of the cone is

$$\psi : I \times \mathbb{R}_+ \to \mathbb{R}^3, \qquad \psi(u,v) = v\gamma(u).$$

(b) The adapted basis of the tangent plane at $\psi(u,v)$ is

$$b_1 = \partial_u \psi = v\dot{\gamma}(u), \qquad b_2 = \partial_v \psi = \gamma(u).$$

Because $\|\gamma(u)\| = 1$ and $\dot{\gamma}(u) \perp \gamma(u)$, we have

$$g_{12} = \langle b_1, b_2 \rangle = \langle v\dot{\gamma}, \gamma \rangle = 0, \qquad g_{11} = \|b_1\|^2 = v^2, \quad g_{22} = \|b_2\|^2 = 1.$$

Hence the metric tensor is

$$G(u,v) = \begin{pmatrix} v^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{so } ds^2 = v^2 du^2 + dv^2.$$

(c) The above is exactly the metric of the Euclidean plane in polar coordinates (replace u by θ and v by r). Thus the cone (minus the vertex) is locally isometric to the plane; explicit local isometry is given by $(u,v) \mapsto (x = v\cos u,\ y = v \sin u)$.

11.3 Let $S \subset \mathbb{R}^3$ be a C^2 surface and $\gamma : I \to S$ be a geodesic.

- **(a)** Recall the definition of a geodesic. Show that the speed of γ is constant.
- **(b)** Show that if S is a sphere, then (nonconstant) geodesics are the great circles parametrized at constant speed. *Hint: You can assume that S is a sphere centered at the origin. Show that, in this case, the vector $\gamma(t)$ is parallel to the normal of S at the point $\gamma(t)$. Use also the relation that $\langle \gamma(t), \gamma(t) \rangle$ is constant (since $\gamma(t) \in S$); you might want to differentiate this relation a few times.*
- **(c)** Consider the planar curve $\sigma : v \to (r(v), z(v))$, $a \leqslant v \leqslant b$, with $\min_{v \in [a,b]} r(v) > 0$, and let S be the surface of revolution obtained by rotating σ around the z axis. In particular, S is parametrized by $\psi : [a,b] \times [0, 2\pi] \to S$, $\psi(u,v) = \big(r(v)\cos(u), r(v)\sin(u), z(v)\big)$. We call the lines $u = const$ *meridians* and the lines (circles) $v = const$ *parallels*.
  - $*$ Show that if γ is a meridian of S and γ is traversed at constant speed, then γ is a geodesic.
  - $*$ Under which condition is a parallel of a surface of revolution a geodesic?

Solution.

(a) If γ is a geodesic then $\ddot{\gamma}(t)$ is orthogonal to every tangent vector of the surface at $\gamma(t)$, in particular $\langle \ddot{\gamma}, \dot{\gamma} \rangle = 0$. Hence

$$\frac{d}{dt}\|\dot{\gamma}(t)\|^2 = \frac{d}{dt}\langle \dot{\gamma}, \dot{\gamma} \rangle = 2\langle \ddot{\gamma}, \dot{\gamma} \rangle = 0,$$

so the speed $\|\dot{\gamma}(t)\|$ is constant.

(b) Assume the sphere is centered at the origin, so it is of the form

$$S = \{x : \quad \|x\| = R\}.$$

Note that, at any $x \in S$, the vector x is parallel to the normal n of S at x. This is because if $\gamma : (-\epsilon, \epsilon) \to S$ is any curve such that $\gamma(0) = x$, then, since $\gamma(t) \in S$, we have $\langle \gamma(t), \gamma(t) \rangle = R^2 =$ const and, therefore, by differentiating: $\langle \dot\gamma, \gamma \rangle = 0$. Evaluated at 0 this means that $x = \gamma(0)$ is perpendicular to $\dot\gamma(0)$. Since this is true for every such curve γ, we infer that x is perpendicular to any tangent vector of S at x.

Let now $\gamma : I \to S$ be a geodesic on the sphere S. By definition, this means that $\ddot\gamma$ is perpendicular to S, i.e. there exists $\lambda : I \to \mathbb{R}$ such that

$$\ddot\gamma(t) = \lambda(t)\gamma(t).$$

We will first show that λ is constant. Starting from the relation $\langle \gamma, \gamma \rangle = R^2$ and differentiating it twice (using also the fact that $\langle \dot\gamma, \dot\gamma \rangle =$ const, setting $\Lambda \doteq \|\dot\gamma\|$), we obtain:

$$0 = \frac{d^2}{dt^2}\langle \gamma, \gamma \rangle = 2\langle \ddot\gamma, \gamma \rangle + 2\langle \dot\gamma, \dot\gamma \rangle = 2\lambda\langle \gamma, \gamma \rangle + 2\Lambda^2 \quad \Rightarrow \quad \lambda(t) = -\frac{\Lambda^2}{R^2} = \text{const}.$$

Thus, γ satisfies

$$\ddot\gamma(t) + \frac{\Lambda^2}{R^2}\gamma(t) = 0.$$

The solution of the above equation is a great circle. To see this, fix $t_0 \in I$. Note that $\dot\gamma(t_0) \perp \gamma(t_0)$ (this is true for any curve on the sphere) and $\|\dot\gamma(t_0)\| = \Lambda$ (from our definition of Λ). By rotating the sphere, we can assume without loss of generality that $\gamma(t_0) = (R, 0, 0)$ and $\dot\gamma(t_0) = (0, \Lambda, 0)$. Then, if $\gamma(t) = (x(t), y(t), z(t))$, the above ODE yields the following triplet of initial value problems:

$$\begin{cases} \ddot x(t) + \frac{\Lambda^2}{R^2}x(t) = 0, \\ x(t_0) = R, \\ \dot x(t_0) = 0, \end{cases} \qquad \begin{cases} \ddot y(t) + \frac{\Lambda^2}{R^2}y(t) = 0, \\ y(t_0) = 0, \\ \dot y(t_0) = \Lambda, \end{cases} \qquad \begin{cases} \ddot z(t) + \frac{\Lambda^2}{R^2}z(t) = 0, \\ z(t_0) = 0, \\ \dot z(t_0) = \Lambda. \end{cases}$$

These initial value problems have unique solutions, which are $x(t) = R\cos\left(\frac{\Lambda}{R}(t - t_0)\right)$, $y(t) = R\sin\left(\frac{\Lambda}{R}(t - t_0)\right)$ and $z(t) = 0$. This is a constant speed parametrization of the great circle $\{x^2 + y^2 = R^2\} \cap \{z = 0\}$.

An alternative way of showing the above: Set $m = \gamma \times \dot\gamma$. Then $m \neq 0$ (since γ and $\dot\gamma$ are nonzero and orthogonal) and

$$\dot m = \dot\gamma \times \dot\gamma + \gamma \times \ddot\gamma = \gamma \times \ddot\gamma = 0,$$

because $\ddot\gamma$ is normal to the sphere and thus colinear to γ. Hence m is constant and $\gamma(t)$ lies in the plane $\{x \mid \langle x, m \rangle = 0\}$ through the origin. Therefore γ is a great circle.

(c) Let S be a surface of revolution, for instance generated by rotating $y = f(x)$ around the x-axis; in this case, a parametrization of the surface is given by $\Psi(u, v) = (v, f(v)\cos(u), f(v)\sin(u))$. The meridians (the lines $u =$ const) are obtained by intersection of S with planes containing the axis of revolution. Therefore, with respect to any parametrization $\gamma(t)$ of a meridian, the acceleration vector $\ddot{\gamma}(t)$ has to be contained in the same plane (i.e. in the plane defined by $\gamma(t)$ and the axis of revolution); this is true in general for any curve that is entirely contained in a plane). Note that, for a surface of revolution, that plane also contains $n(\gamma(t))$ (since $n(u, v)$ is parallel to $\partial_u\Psi \times \partial_v\Psi = (f(v)f'(v), f(v)\cos(u), f(v)\sin(u)))$. In particular, this plane is spanned by $n(\gamma(t))$ and $T_\gamma(t)$. In the case when the parametrization is chosen to be of constant speed, we must have that $\ddot{\gamma} \perp \dot{\gamma}$. Therefore, in this case, $\ddot{\gamma}(t)$ is contained in the span of $\{n(\gamma(t)), T_\gamma(t)\}$ and is perpendicular to $T_\gamma(t)$, therefore it has to be parallel to $n(\gamma(t))$. Hence, any meridian parametrized with constant speed is a geodesic.

A parallel (a circle at fixed meridional parameter, $\{v =$ const) can be parametrized by $\gamma(u) = (v, f(v)\cos(u), f(v)\sin(u))$, which is a constant speed parametrization. In this case, the acceleration $\ddot{\gamma}(u) = (0, -f(v)\cos(u), -f(v)\sin(u))$ is parallel to the normal (which we computed above is parallel to $(f(v)f'(v), f(v)\cos(u), f(v)\sin(u)))$ if and only if $f'(v) = 0$, i.e. at points where the derivative of the generating profile vanishes. Note that at all other points, there is no reparametrization of the parallel that can turn it into a geodesic; any reparametrization would only generate a component of $\ddot{\gamma}$ that is parallel to $\dot{\gamma}(u) = (0, -f(v)\sin u, f(v)\cos u)$, hence it would not be able to cancel out the x component of n.

11.4 Let $H \subset \mathbb{R}^3$ be the helicoid given by the equation $x \sin z = y \cos z$ and let $C \subset \mathbb{R}^3$ be the right circular cylinder $x^2 + y^2 = 1$.

(a) Show that the intersection of these two surfaces is the disjoint union of the images of the two helices $\gamma_\pm : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma_+(t) = (\cos t, \sin t, t), \qquad \gamma_-(t) = (-\cos t, -\sin t, t).$$

That is,

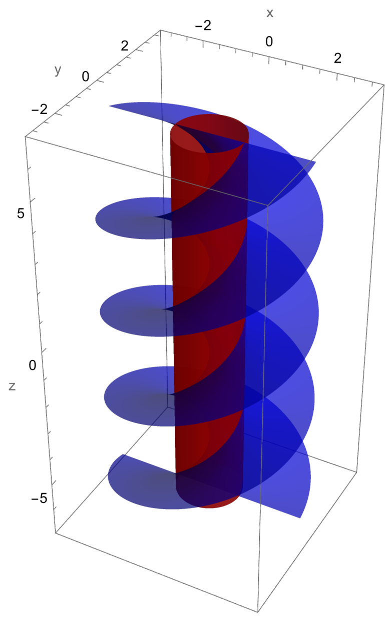
$$H \cap C = \gamma_+(\mathbb{R}) \cup \gamma_-(\mathbb{R}) \quad \text{and} \quad \gamma_+(\mathbb{R}) \cap \gamma_-(\mathbb{R}) = \emptyset.$$

(In particular $H \cap C$ has two connected components.)

(b) Is $\gamma_\pm(t)$ a geodesic of the cylinder?

(c) Is $\gamma_\pm(t)$ a geodesic of the helicoid?

Solution.



(a) For any $t \in \mathbb{R}$, $\gamma_+(t)$ satisfies the relations defining both H and C

$$x \sin z - y \cos z = \cos t \sin t - \sin t \cos t = 0,$$

and

$$\cos^2 t + \sin^2 t = 1,$$

therefore $\gamma_+(t) \in H \cap C$. The same holds for γ_-. Conversely, if $(x, y, z) \in H \cap C$ then $x \sin z = y \cos z$, so there exists $\lambda \in \mathbb{R}$ with $x = \lambda \cos z$, $y = \lambda \sin z$. But $x^2 + y^2 = 1$ forces $\lambda = \pm 1$, hence the point is on γ_+ or γ_-. These curves are disjoint since we cannot have $\gamma_+(u) = \gamma_-(v)$ for any $u, v \in \mathbb{R}$; this would imply the equalities $\cos u = -\cos v$, $\sin u = -\sin v$ and $u = v$, which cannot hold simultaneously.

(b) Yes. For the cylinder $x^2 + y^2 = 1$ one has $\nabla f = (2x, 2y, 0)$ with $f(x, y, z) = x^2 + y^2 - 1$. For $\gamma_+(t)$,

$$\ddot{\gamma}_+(t) = (-\cos t, -\sin t, 0), \qquad \nabla f(\gamma_+(t)) = (2\cos t, 2\sin t, 0),$$

which are colinear. Hence $\ddot{\gamma}_+$ is normal to the cylinder and γ_+ is a geodesic of the cylinder. Same for γ_-.

(c) No. For the helicoid $g(x, y, z) = x \sin z - y \cos z$ we have

$$\nabla g(x, y, z) = (\sin z, -\cos z, x \cos z + y \sin z).$$

Evaluating along $\gamma_+(t)$ yields $\nabla g(\gamma_+(t)) = (\sin t, -\cos t, 1)$ which is not colinear with $\ddot{\gamma}_+(t) = (-\cos t, -\sin t, 0)$ (indeed they are orthogonal). Thus γ_+ is not a geodesic of the helicoid. Same for γ_-.

11.5 Compute explicitly the Gauss map $\nu : E \to S^2$ of the ellipsoid given implicitly by

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\},$$

with $a, b, c \neq 0$. Provide a formula for $\nu(x, y, z)$ for each point $(x, y, z) \in E$. What do you observe in the case $a = b = c = 1$ (i.e. when E is the unit sphere)?

Solution. Write $E = \{f = 1\}$ with $f(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Then the Gauss map (up to sign depending on chosen co-orientation) is

$$\nu(x, y, z) = \pm \frac{\nabla f}{\|\nabla f\|} = \pm \frac{\left(x/a^2,\ y/b^2,\ z/c^2\right)}{\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}}.$$

If $a = b = c = 1$ (unit sphere) this returns $\nu(x, y, z) = \pm(x, y, z)/\|(x, y, z)\| = \pm(x, y, z)$ on the sphere, so the Gauss map is either the identity or the antipodal map depending on the chosen orientation.

11.6 Let $\gamma : I \to S$ be a regular C^2 curve drawn on a co-oriented regular surface $S \subset \mathbb{R}^3$. The Darboux frame along γ relative to the surface S is the orthonormal moving frame $\{T_\gamma(t), \mu(t), n(t)\}$ where $T_\gamma(t) = \dfrac{\dot\gamma(t)}{V_\gamma(t)}$ is the unit tangent to γ, $n(t)$ is the Gauss map of S evaluated at $\gamma(t)$, and $\mu(t) = n(t) \times T_\gamma(t)$.

Denote by $K_\gamma(t)$ the curvature vector of γ. Recall that the normal curvature and the geodesic curvature of γ are the functions

$$k_n(t) = \langle K_\gamma(t), n(t)\rangle \quad \text{and} \quad k_g(t) = \langle K_\gamma(t), \mu(t)\rangle.$$

(a) Show that $\kappa(t)^2 = k_n(t)^2 + k_g(t)^2$, where κ is the curvature of γ (as a curve in $\mathbb{R}^3$).

(b) Prove that γ is a geodesic if and only if its speed is constant and its geodesic curvature vanishes.

(c) Compute the Darboux frame, the geodesic curvature and the normal curvature of the small circle on the unit sphere $\mathbb{S}^2$ defined by the equations $x^2 + y^2 + z^2 = 1$ and $z = c$ (where $-1 < c < 1$).

Solution.

(a) Since $\{T_\gamma(t), \mu_\gamma(t), n(\gamma(t)\} \doteq \{T(t), \mu(t), n(t)\}$ is an orthonormal frame, we can decompose K_γ into its orthogonal components:

$$K_\gamma = \langle K_\gamma, T\rangle T + \langle K_\gamma, \mu\rangle \mu + \langle K_\gamma, n\rangle n = .$$

But $\langle K, T\rangle = 0$ (curvature vector is orthogonal to the tangent) and $\langle K_\gamma, \mu\rangle = k_g$, $\langle K_\gamma, n\rangle = k_n$, hence

$$\|K\|^2 = \langle K, \nu\rangle^2 + \langle K, \mu\rangle^2 = k_n^2 + k_g^2,$$

namely $\kappa^2 = k_n^2 + k_g^2$.

(b) By definition γ is a geodesic iff $\ddot{\gamma}(t)$ is orthogonal to the tangent plane $T_{\gamma(t)}S$ for all t. Decomposing $\ddot{\gamma} = \dot{V}T + V_\gamma^2 K_\gamma$ and using the expression for K_γ computed earlier, we get

$$\ddot{\gamma} = \dot{V}T + k_g\mu + k_n n.$$

Thus, the tangential components to the surface vanish iff $\dot{V} = 0$ and $k_g = 0$. Thus γ is a geodesic iff its speed is constant and $k_g \equiv 0$.

(c) Choose the exterior co-orientation on S^2; then the Gauss map ν is the identity $n(x) = x$. The small circle at height $z = c$ has radius $a = \sqrt{1-c^2}$; parametrize it by

$$\gamma(t) = (a\cos t, a\sin t, c), \qquad V_\gamma = \|\dot{\gamma}\| = a.$$

Then

$$T(t) = \frac{\dot{\gamma}}{V_\gamma} = (-\sin t, \cos t, 0), \qquad K(t) = \frac{1}{V_\gamma}\dot{T}(t) = -\frac{1}{a}(\cos t, \sin t, 0).$$

Compute the Darboux frame:

$$n(t) = \gamma(t) = (a\cos t, a\sin t, c), \quad \mu(t) = n \times T = (-c\cos t, -c\sin t, a).$$

Hence

$$k_n(t) = \langle K(t), n(t)\rangle = -1, \qquad k_g(t) = \langle K(t), \mu(t)\rangle = \frac{c}{a}.$$

(One can check $k_n^2 + k_g^2 = 1/a^2 = \|K\|^2$ as a consistency check.) Note: the normal curvature may be negative depending on chosen co-orientation; choosing the inward normal would flip signs.

11.7 Continuing with the notation of the previous exercise, define the geodesic torsion of γ by

$$\tau_g(t) = \frac{1}{V_\gamma(t)}\langle \dot{n}(t), \mu(t)\rangle.$$

- **(a)** Compute the geodesic torsion of the small circle on $\mathbb{S}^2$ defined by $z = c$.
- **(b)** Prove that the Darboux frame satisfies the following differential equations:

$$\begin{cases} \frac{1}{V_\gamma}\dot{T} = k_g\,\mu + k_n\,n, \\ \frac{1}{V_\gamma}\dot{n} = -k_n\,T + \tau_g\,\mu, \\ \frac{1}{V_\gamma}\dot{\mu} = -k_g\,T - \tau_g\,n. \end{cases}$$

Solution.

(a) For the small circle we have $n(t) = \gamma(t) = (a\cos t, a\sin t, c)$ and $\mu(t) = (-c\cos t, -c\sin t, a)$ (up to orientation). Then $\dot{n} = \dot{\gamma} = (-a\sin t, a\cos t, 0)$. Since $V_\gamma = a$, the geodesic torsion is

$$\tau_g(t) = \frac{1}{a}\langle \dot{\nu}, \mu\rangle = 0.$$

So the geodesic torsion of a small circle on a sphere is zero.

(b) These formulas are entirely analogous to the Serret–Frenet equations. The first equation follows from the decomposition of the curvature vector $K = \frac{1}{V}\dot{T}$ into its components along μ and ν. The second and third equations come from expanding $\dot{\nu}$ and $\dot{\mu}$ in the Darboux frame and using the identities

$$\langle \dot{\nu}, \nu\rangle = 0, \quad \langle \dot{\nu}, T\rangle = -\langle \nu, \dot{T}\rangle = -Vk_n, \quad \langle \dot{\nu}, \mu\rangle = V\tau_g,$$

and similarly for $\dot{\mu}$.